

TECHNICAL REPORT 1934  
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# **Matrix Dilations via Cosine–Sine Decomposition**

J. C. Allen  
D. Arceo

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SSC San Diego

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## EXECUTIVE SUMMARY

This report originated in the  $H^\infty$  Research Initiative of the Office of Naval Research and the In-House Laboratory Independent Research (ILIR) Program of Space and Naval Warfare Systems Center San Diego (SSC San Diego). These programs focused on  $H^\infty$  engineering for fleet applications—wideband impedance matching and wideband amplifier optimization. Research in these applications produced several papers [33], [32], [2], [3], four patents, a book [1], and sparked the Defense Advanced Research Projects Agency’s interest in digital  $H^\infty$  engineering.

$H^\infty$  engineering computes the best possible performance bounds. For example, a wideband antenna should be matched to the line impedance to minimize power and prevent amplifier burnout. The circuit designer matches the antenna by searching for a lossless 2-port that minimizes the Voltage Standing Wave Ratio (VSWR). Traditionally, the circuit designer guesses a 2-port topology and then optimizes over its circuit elements. This process is repeated over various topologies, hoping that a 2-port that is “good enough” turns up. In contrast,  $H^\infty$  engineering computes the smallest VSWR attainable by any lossless matching 2-port independent of circuit topology [21]. This best VSWR provides an absolute benchmark to assess candidate circuits. and brings some order to matching-circuit selection. Moreover, the  $H^\infty$  methods also produce some information on an optimal 2-port circuit. If

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

is the unknown scattering matrix of an optimal 2-port, then the  $H^\infty$  methods *compute*  $s_{11}$ . Consequently, techniques that *dilate* the passive 1-port  $s_{11}$  into a lossless 2-port solve the synthesis problem [5].

Dilations are basic to electrical engineering, signal processing, and operator theory. This report makes explicit the algebraic structure of these matrix dilations by the Cosine-Sine Decomposition (CSD). The CSD provides a unifying computational framework for parameterizing all unitary dilations of a given matrix, parameterizing all contractive dilations, and generalizes to produce all  $J$ -unitary dilations. These dilations are foundational for the synthesis problem. Moreover, the CSD applies to several signal-processing problems [29], [35], [16]. Consequently, the CSD is a simple and flexible technique that can be applied to theoretical and computational dilation problems.

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# 1 Matrix Dilations

Matrix dilations can be approached in several ways. The historical approach could start with Halmos' original work on unitary and normal dilations [17]. The harmonic analysis approach could start from the work of Nagy & Foiaş [13] that links analytic functions and dilations. Another approach follows from the link between operators and electrical circuits discovered by Helton [19]; in this approach, circuit synthesis is equivalent to either unitary [28] or  $J$ -unitary matrix dilations [20], [21]. These approaches are only a few ways to access the massive literature on dilation theory. This report approaches dilation theory using the *Cosine-Sine decomposition* (CSD). The focus is on matrix dilations so the algebraic patterns clearly stand out.

Let  $A$  be a given  $M \times M$  matrix. The dilation problem is to parameterize all *unitary dilations*  $U_A$  of matrix  $A$ : find all matrices  $B \in \mathbb{C}^{M \times N}$ ,  $C \in \mathbb{C}^{N \times M}$ , and  $D \in \mathbb{C}^{N \times N}$  so that the dilation

$$U_A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is unitary:

$$U_A^H U_A = \begin{bmatrix} I_M & 0 \\ 0 & I_N \end{bmatrix}.$$

Here  $I_M$  denotes the  $M \times M$  identity matrix and the superscript  $H$  denotes the Hermitian or conjugate transpose. The CSD parameterizes all unitary dilations of  $A$  and reveals how the dilation of smallest size is encoded in  $A$ .

A unitary dilation is not the only type of dilation that can be considered. A significant generalization replaces the unitary equality

$$U_A^H U_A = I_{M+N}$$

with the contractive inequality

$$U_A^H U_A \leq I_{M+N}.$$

In the latter case,  $U_A$  is called a *contractive dilation* of  $A$ . The CSD parameterizes all these contractive dilations. The simple patterns give a short proof of Parrot's Theorem, which has connections to  $H^\infty$  theory and electrical engineering. It is in electrical engineering that the CSD generalizes to its  $J$ -unitary version. In this setting, the problem is to find all dilations of  $A$  that are  $J$ -unitary or hyperbolic:

$$U_A^H J U_A = J := \begin{bmatrix} I_M & 0 \\ 0 & -I_N \end{bmatrix}.$$

Electrical engineers routinely use a natural mapping between unitary matrices and hyperbolic matrices. Under this mapping, the CSD naturally turns into the Hyperbolic Cosh-Sinh decomposition (HCSD). From this HCSD, all  $J$ -unitary dilations are obtained. Thus, the CSD admits a hyperbolic generalization with engineering applications. We conclude by pointing out that several other matrix decompositions also generalize to hyperbolic versions with signal-processing applications.

## 2 Notation

The set of all complex-valued  $M \times N$  matrices is denoted by  $\mathbb{C}^{M \times N}$ . If  $X \in \mathbb{C}^{M \times N}$ ,  $X$  is said to have size  $M \times N$ , which will be denoted  $X \sim M \times N$ . The Hermitian or conjugate transpose of  $X$  is denoted by  $X^H$ . The group of  $M \times M$  unitary matrices is denoted by

$$\mathcal{U}(M) := \{U \in \mathbb{C}^{M \times M} : U^H U = I_M\}.$$

A diagonal matrix  $\Theta$  is denoted by

$$\Theta = \text{diag}(\theta_1, \dots, \theta_M) = \begin{bmatrix} \theta_1 & 0 & \dots & 0 \\ 0 & \theta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \theta_M \end{bmatrix}.$$

With a slight abuse of notation,  $\cos(\Theta)$  denotes the diagonal matrix

$$\cos(\Theta) = \begin{bmatrix} \cos(\theta_1) & 0 & \dots & 0 \\ 0 & \cos(\theta_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \cos(\theta_M) \end{bmatrix}.$$

Similarly,  $\sin(\Theta)$  denotes the diagonal matrix

$$\sin(\Theta) = \begin{bmatrix} \sin(\theta_1) & 0 & \dots & 0 \\ 0 & \sin(\theta_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sin(\theta_M) \end{bmatrix}.$$

## 3 Cosine-Sine Decomposition

If a matrix  $A$  is square ( $N = M$ ), one unitary dilation of  $A$  is the Halmos dilation [17, Chapter 23]:

$$U_A = \begin{bmatrix} A & -(I_M - AA^H)^{1/2} \\ (I_M - A^H A)^{1/2} & A^H \end{bmatrix},$$



where the positive semidefinite square root is selected. Halmos points out that this dilation has a nice geometric interpretation. If  $A$  has a singular-value decomposition  $A = U \cos(\Theta) V^H$ , where  $U$  and  $V$  are unitary matrices, then  $U_A$  generalizes the classic rotation matrix as follows:

$$U_A = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \cos(\Theta) & -\sin(\Theta) \\ \sin(\Theta) & \cos(\Theta) \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}^H.$$

This is a special case of the CSD.

**Theorem 1 (CSD)** [34, page 37], [15, page 77] *Let the unitary matrix  $W \in \mathcal{U}(M+N)$  be partitioned as*

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{matrix} \mathbb{C}^M \\ \mathbb{C}^N \end{matrix}.$$

*If  $N \geq M$ , then there are unitary matrices  $U_{11}, V_{11} \in \mathcal{U}(M)$  and unitary matrices  $U_{22}, V_{22} \in \mathcal{U}(N)$  such that*

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} \left[ \begin{array}{c|cc} C & -S & 0 \\ \hline S & C & 0 \\ 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}^H,$$

*where  $C \geq 0$  and  $S \geq 0$  are diagonal matrices satisfying  $C^2 + S^2 = I_M$ .*

Thus, the CSD simultaneously produces the *singular-value decompositions* (SVDs) for  $W_{11}$ ,  $W_{21}$ ,  $W_{12}$ , and  $W_{22}$  from the sine and cosine matrices. The converse parameterizes the unitary dilations of  $A$ .

## 4 Parameterizing Unitary Dilations

Given the SVD of a matrix  $A$ , the CSD shows us how to get a unitary dilation of  $A$ . The utility of the CSD is that all unitary dilations of  $A$  with  $N \geq M$  can be obtained this way.

**Corollary 1** [23, Problem 1.6.21] *Let  $A \in \mathbb{C}^{M \times M}$  be a contraction. Select any singular-value decomposition*

$$A = U_{11} \cos(\Theta) V_{11}^H,$$

where  $\Theta = \text{diag}(\theta_1, \dots, \theta_M)$  is sorted in ascending order:  $0 \leq \theta_1 \leq \dots \leq \theta_M \leq \pi/2$ . If  $N \geq M$ , then all unitary dilations of  $A$  with  $D \in \mathbb{C}^{N \times N}$  are parameterized as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} \left[ \begin{array}{c|c|c} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ \hline 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}^H,$$

where  $U_{22}, V_{22} \in \mathcal{U}(N)$ .

**Proof:** If  $U_A$  has the given CSD, then  $U_A$  is a unitary dilation of  $A$  and demonstrates that unitary dilations of  $A$  exist for  $N \geq M$ . Conversely, if  $U_A$  is a unitary dilation of  $A$  with  $N \geq M$ , the CSD gives

$$U_A = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix} \left[ \begin{array}{c|c|c} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ \hline 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{bmatrix}^H,$$

where  $X_{11}, Y_{11} \in \mathcal{U}(M)$ ,  $X_{22}, Y_{22} \in \mathcal{U}(N)$ , and  $A = X_{11} \cos(\Theta) Y_{11}^H$ . The unicity of the SVD allows us to set  $X_{11} = U_{11}$  and  $Y_{11} = V_{11}$  and use only  $U_{22}$  and  $V_{22}$  to parameterize the unitary dilations. To demonstrate this claim, observe that  $A$  has the SVDs

$$A = U_{11} \cos(\Theta) V_{11}^H = X_{11} \cos(\Theta) Y_{11}^H.$$

So, how unique are  $U_{11}$  and  $V_{11}$  compared to  $X_{11}$  and  $Y_{11}$ ? Write

$$\cos(\Theta) = \begin{bmatrix} c_1 I_{m_1} & & & & \\ & c_2 I_{m_2} & & & \\ & & \ddots & & \\ & & & c_K I_{m_K} & \\ & & & & 0 \end{bmatrix},$$

where  $c_1 > c_2 > \dots > c_K > 0$  are the distinct non-zero singular values of multiplicity  $m_k$ . By [23, Theorem 3.1.1'], there are unitary matrices  $W_1, W_2, \dots, W_K, E_1$ , and  $E_2$  such that

$$F = \begin{bmatrix} W_1 & & & & \\ & W_2 & & & \\ & & \ddots & & \\ & & & W_K & \\ & & & & E_1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} W_1 & & & & \\ & W_2 & & & \\ & & \ddots & & \\ & & & W_K & \\ & & & & E_2 \end{bmatrix}$$

link the two SVDs of  $A$  as  $X_{11} = U_{11} F$  and  $Y_{11} = V_{11} G$ . By construction, both  $F$  and  $G$  commute with  $\cos(\Theta)$  and  $\sin(\Theta)$ , and satisfy

$$\cos(\Theta) = F \cos(\Theta) G^H = G^H \cos(\Theta) F.$$

Substitution gives

$$\begin{aligned}
U_A &= \begin{bmatrix} U_{11}F & 0 \\ 0 & X_{22} \end{bmatrix} \left[ \begin{array}{c|c|c} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ \hline 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} G^H V_{11}^H & 0 \\ 0 & Y_{22}^H \end{bmatrix} \\
&= \begin{bmatrix} U_{11} & 0 \\ 0 & X_{22} \end{bmatrix} \left[ \begin{array}{c|c|c} F \cos(\Theta) G^H & -F \sin(\Theta) & 0 \\ \sin(\Theta) G^H & \cos(\Theta) & 0 \\ \hline 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} V_{11}^H & 0 \\ 0 & Y_{22}^H \end{bmatrix} \\
&= \begin{bmatrix} U_{11} & 0 \\ 0 & X_{22} \end{bmatrix} \left[ \begin{array}{c|c|c} \cos(\Theta) & -\sin(\Theta)F & 0 \\ G^H \sin(\Theta) & G^H \cos(\Theta)F & 0 \\ \hline 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} V_{11}^H & 0 \\ 0 & Y_{22}^H \end{bmatrix} \\
&= \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} \left[ \begin{array}{c|c|c} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ \hline 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} V_{11}^H & 0 \\ 0 & V_{22}^H \end{bmatrix},
\end{aligned}$$

where

$$U_{22} = X_{22} \begin{bmatrix} G^H & 0 \\ 0 & I_{N-M} \end{bmatrix}, \quad V_{22} = Y_{22} \begin{bmatrix} F^H & 0 \\ 0 & I_{N-M} \end{bmatrix}.$$

Because  $X_{22}, Y_{22} \in \mathcal{U}(N)$  are arbitrary,  $U_{22}$  and  $V_{22}$  are also arbitrary. Thus, we may fix  $U_{11}$  and  $V_{11}$  to parameterize  $U_A$  using only  $U_{22}$  and  $V_{22} \in \mathcal{U}(N)$ . ///

The proof shows how the non-uniqueness in the SVD of  $A$  (the matrices  $F$  and  $G$ ) can be peeled off and then cast into the arbitrary  $U_{22}$  and  $V_{22}$  matrices. Thus,  $U_{11}$  and  $V_{11}$  may be fixed so that the dilations are parameterized only by  $U_{22}, V_{22} \in \mathcal{U}(N)$ .

For dilations with  $N \leq M$ , consider the following numerical example. Partition the random unitary matrix  $U_A$  as follows:

$$U_A = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[ \begin{array}{cccc|cc} -0.3944 & -0.2202 & 0.1686 & 0.0281 & -0.5633 & 0.4974 & 0.4495 \\ -0.5276 & -0.0880 & -0.5915 & 0.1517 & 0.1399 & 0.3175 & -0.4697 \\ 0.6000 & -0.0909 & -0.2244 & 0.6823 & -0.2040 & 0.2708 & -0.0318 \\ -0.0484 & 0.1497 & -0.3991 & -0.0607 & -0.7027 & -0.5589 & -0.0777 \\ 0.3307 & -0.7570 & -0.2090 & -0.5057 & -0.0567 & 0.0589 & -0.1069 \\ \hline -0.2331 & -0.4216 & -0.2224 & 0.3707 & 0.3144 & -0.4437 & 0.5342 \\ 0.2001 & 0.4028 & -0.5647 & -0.3375 & 0.1602 & 0.2563 & 0.5230 \end{array} \right].$$

Corollary 1 would dilate  $A$  to a  $10 \times 10$  unitary matrix. However,  $A$  has the singular-value matrix

$$\begin{bmatrix} I_3 & 0 \\ 0 & \cos(\Theta) \end{bmatrix} = \left[ \begin{array}{c|c} 1.0000 & \\ \hline 1.0000 & 0.8771 \\ \hline & 0.2452 \end{array} \right].$$

The two smallest singular values clue us that the  $5 \times 5$  matrix  $A$  came from the  $7 \times 7$  unitary matrix  $U_A$ . More generally, the number of singular values that are strictly less than 1 actually encode the size of the smallest unitary dilation.

**Corollary 2** [23, Problem 1.6.21] Let  $A \in \mathbb{C}^{M \times M}$  be a contraction with SVD

$$A = U_{11} \begin{bmatrix} I_L & 0 \\ 0 & \cos(\Theta) \end{bmatrix} V_{11}^H,$$

where  $\Theta = \text{diag}(\theta_1, \dots, \theta_K)$  and  $0 < \theta_1 \leq \dots \leq \theta_K \leq \pi/2$ . If  $N \leq M$ , then all unitary dilations of  $A$  with  $D \sim N \times N$  must have  $N \geq K$  and take the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} \left[ \begin{array}{cc|c} I_{M-N} & 0 & 0 \\ 0 & \cos(\Psi) & -\sin(\Psi) \\ \hline 0 & \sin(\Psi) & \cos(\Psi) \end{array} \right] \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}^H,$$

where  $U_{22}, V_{22} \in \mathcal{U}(N)$  and  $\Psi = \begin{bmatrix} 0_{N-K} & 0 \\ 0 & \Theta \end{bmatrix}$ .

**Proof:** The CSD permits us to select the decomposition

$$\begin{bmatrix} D & C \\ B & A \end{bmatrix} = \begin{bmatrix} X_{22} & 0 \\ 0 & X_{11} \end{bmatrix} \left[ \begin{array}{cc|c} \cos(\Psi) & \sin(\Psi) & 0 \\ -\sin(\Psi) & \cos(\Psi) & 0 \\ \hline 0 & 0 & I_{M-N} \end{array} \right] \begin{bmatrix} Y_{22} & 0 \\ 0 & Y_{11} \end{bmatrix}^H,$$

where  $\Psi = \text{diag}(\psi_1, \dots, \psi_N)$ ;  $0 \leq \psi_1 \leq \dots \leq \psi_N \leq \pi/2$ ;  $X_{22}, Y_{22} \in \mathcal{U}(N)$ ;  $X_{11}, Y_{11} \in \mathcal{U}(M)$ . By some non-trivial relabeling, we may write

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix} \left[ \begin{array}{cc|c} I_{M-N} & 0 & 0 \\ 0 & \cos(\Psi) & -\sin(\Psi) \\ \hline 0 & \sin(\Psi) & \cos(\Psi) \end{array} \right] \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{bmatrix}^H. \quad (1)$$

By assumption,  $A$  has  $K$  singular values strictly less than 1 and  $L$  singular values exactly equal to 1. Then  $K + L = M$  and  $M - N \leq L$ . Then all dilations must have  $N \geq M - L = K$ . The unicity of the singular values [23, page 146] and the ordering of the  $\psi_n$ 's permit us to write  $\Psi$  as stated. When  $N = K$ , it follows that  $\Psi = \Theta$  so that the smallest dilations exist. Equation 1 now permits the application of the unicity arguments from Corollary 1. Thus, we may fix  $X_{11}$  and  $Y_{11}$  as the unitary matrices  $U_{11}$  and  $V_{11}$  from the SVD of  $A$  and sweep out all dilations of  $A$  by sweeping over  $X_{22}, Y_{22} \in \mathcal{U}(N)$ . ///

The important special case of Corollary 2 is the parameterization of the unitary dilations of smallest size:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} \left[ \begin{array}{cc|c} I_L & 0 & 0 \\ 0 & \cos(\Theta) & -\sin(\Theta) \\ \hline 0 & \sin(\Theta) & \cos(\Theta) \end{array} \right] \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}^H,$$

where  $|\cos(\theta_m)| < 1$ . This case explains the preceding numerical example where  $A \sim 5 \times 5$  had  $I_L \sim 3 \times 3$  and  $\Theta \sim 2 \times 2$ . Thus,  $A$  has unitary dilations of size  $7 \times 7$  or larger.

## 5 Parameterizing Contractive Dilations

Given a matrix  $A$  that is a contraction, the problem is to find all dilations

$$T_A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

that are also contractions:  $T_A^H T_A \leq I_{M+N}$ . The CSD provides a straight-forward parameterization of all the  $T_A$ 's by compressing a unitary dilation.

**Corollary 3** (Adapted from [7], [30], [27], [36], [14, Corollary 3.5]) *Let  $A \in \mathbb{C}^{M \times M}$  be a contraction. Select any singular-value decomposition*

$$A = U_{11} \cos(\Theta) V_{11}^H,$$

where  $\Theta = \text{diag}(\theta_1, \dots, \theta_M)$  is sorted in ascending order:  $0 \leq \theta_1 \leq \dots \leq \theta_M \leq \pi/2$ . Then all contractive dilations of  $A$  with  $D \sim N \times N$  may be parameterized as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} U_{11} & 0 & 0 \\ 0 & U_{22} & U_{23} \end{bmatrix} \begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ 0 & 0 & I_{2N} \end{bmatrix} \begin{bmatrix} V_{11}^H & 0 \\ 0 & V_{22}^H \\ 0 & V_{23}^H \end{bmatrix},$$

where  $[U_{22} \ U_{23}]$ ,  $[V_{22} \ V_{23}] \in P_N \mathcal{U}(M + 2N)$ . Here,  $P_N$  denotes the orthogonal projection onto the first  $N$  components of  $\mathbb{C}^{M+2N}$ .

**Proof:** Let  $P_{M+N}$  denote the orthogonal projection onto the first  $M + N$  components of  $\mathbb{C}^{2M+2N}$ . By Corollary 1,  $A$  has unitary dilations of the form

$$U_A = \begin{bmatrix} U_{11} & 0 & 0 \\ 0 & U_{22} & U_{23} \\ 0 & U_{32} & U_{33} \end{bmatrix} \begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ 0 & 0 & I_{2N} \end{bmatrix} \begin{bmatrix} V_{11}^H & 0 \\ 0 & V_{22}^H & V_{32}^H \\ 0 & V_{23}^H & V_{33}^H \end{bmatrix}.$$

Then  $T_A = P_{M+N} U_A | \mathbb{C}^{M+N}$  is a contractive dilation of  $A$  with  $D \in \mathbb{C}^{N \times N}$ . Conversely, let  $T_A$  be a contractive dilation. By Corollary 1, the unitary Halmos dilation

$$U_{T_A} = \begin{bmatrix} T_A & -(I_{M+N} - T_A T_A^H)^{1/2} \\ (I_{M+N} - T_A^H T_A)^{1/2} & T_A^H \end{bmatrix}$$

admits the factorization

$$U_{T_A} = \begin{bmatrix} U_{11} & 0 & 0 \\ 0 & U_{22} & U_{23} \\ 0 & U_{32} & U_{33} \end{bmatrix} \begin{bmatrix} \cos(\Theta) & -\sin(\Theta) & 0 \\ \sin(\Theta) & \cos(\Theta) & 0 \\ 0 & 0 & I_{2N} \end{bmatrix} \begin{bmatrix} V_{11}^H & 0 \\ 0 & V_{22}^H & V_{32}^H \\ 0 & V_{23}^H & V_{33}^H \end{bmatrix}.$$

Then  $T_A = P_{M+N} U_A | \mathbb{C}^{M+N}$ . ///

This whole apparatus generalizes to operators on Hilbert spaces [14]. A good representative of the CSD in action on a Hilbert space is Parrot's Theorem.

**Theorem 2 (Parrot)** [7], [30], [27], [4] *Let  $\mathcal{H}$  and  $\mathcal{K}$  be Hilbert spaces with orthogonal decompositions  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  and  $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ . Let  $A : \mathcal{H}_1 \rightarrow \mathcal{K}_1$ ,  $B : \mathcal{H}_2 \rightarrow \mathcal{K}_1$ ,  $C : \mathcal{H}_1 \rightarrow \mathcal{K}_2$ , be fixed operators. For  $D : \mathcal{H}_2 \rightarrow \mathcal{K}_2$ , define  $T_D : \mathcal{H} \rightarrow \mathcal{K}$  as the operator*

$$T_D = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{matrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{matrix} \begin{matrix} \\ \end{matrix}.$$

Then

$$\inf\{\|T_D\|\} = \max\left\{\left\|\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\right\|, \left\|\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right\|\right\}. \quad (2)$$

The CSD provides a proof of Parrot's Theorem for the matrix case. Without loss of generality, scale the right side of Equation 2 to 1 so that  $\|T_D\| \geq 1$  for any operator  $D$ . Equality is demonstrated by finding a contractive  $T_D$ . The scaling also forces both matrices in the right side of Equation 2 to be contractions. By Corollary 3, both matrices admit the representations:

$$\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} U_{11} \cos(\Theta) V_{11}^H & 0 \\ U_{22} \sin(\Theta) V_{11}^H & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_{11} \cos(\Theta) V_{11}^H & -U_{11} \sin(\Theta) V_{22}^H \\ 0 & 0 \end{bmatrix}.$$

Combining both representations gives the contraction

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} U_{11} \cos(\Theta) V_{11}^H & -U_{11} \sin(\Theta) V_{22}^H \\ U_{22} \sin(\Theta) V_{11}^H & U_{22} \cos(\Theta) V_{22}^H \end{bmatrix}$$

and proves Parrot's Theorem.

Meinguet [27] offers a fine exposition of Parrot's Theorem and its applications. A key application is Nehari's Theorem [36], [30] that is the foundation of  $H^\infty$  engineering [22]. A link with dilation theory can be traced as follows. By 1970, Nagy & Foias [13] developed a dilation theory for analytic functions whose values are contractions on a Hilbert space. In 1972, J. W. Helton [19] connected their dilation theory to the main realizability theorem of electrical engineering. By 1982, Helton [21] and his colleagues had made deep connections between operator theory, electrical engineering, and control theory. These applications enriched the operator theory with significant generalizations. In particular, the dilation theory admits a nice generalization to dilating  $J$ -unitary or hyperbolic matrices. Electrical engineering problems provide an excellent motivation to consider the hyperbolic matrices and  $J$ -unitary dilations.

## 6 Hyperbolic Matrices in Electrical Engineering

A basic object in electrical engineering is the  $N$ -port. The  $N$ -port is a “black box” with  $N$  pairs of wires sticking out of it. The word “port” means that each pair of wires obeys a conservation of current—the current flowing into one wire of the pair equals the current flowing out of the other wire. Figure 1 shows a 2-port with voltage sources driving each port. The  $N$ -port is the collection of voltage  $\mathbf{v}$  and current  $\mathbf{i}$  vectors that can appear on its ports [20]. Thus, the  $N$ -port is really a subset  $\mathcal{N}$  of a larger voltage-current space, typically an  $L^2$  space.

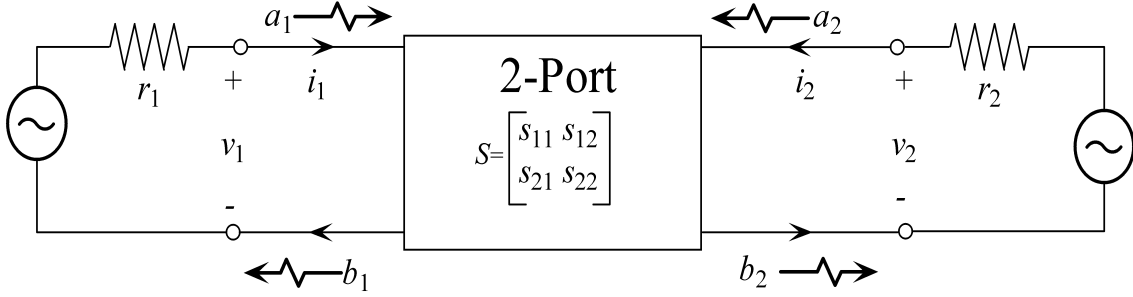


Figure 1: 2-port voltages, currents, and scattering parameters.

Physical assumptions about the  $N$ -port translate into geometric statements about  $\mathcal{N}$ . For example, a linear  $N$ -port is equivalent to the subset  $\mathcal{N}$  being a linear subspace. Under more restrictive assumptions, the  $N$ -port can be the graph of a linear operator. For example, if the  $N$ -port that relates voltage and current as  $\mathbf{v} = Z\mathbf{i}$  is characterized by the impedance matrix  $Z$ :

$$\mathcal{N} = \left\{ \begin{bmatrix} Z\mathbf{i} \\ \mathbf{i} \end{bmatrix} \right\}.$$

Likewise, if  $\mathbf{i} = Y\mathbf{v}$ , the  $N$ -port is characterized by its admittance matrix  $Y$ :

$$\mathcal{N} = \left\{ \begin{bmatrix} \mathbf{v} \\ Y\mathbf{v} \end{bmatrix} \right\}.$$

Not all  $N$ -ports necessarily admit impedance or admittance matrices—classic examples are open circuits, short circuits, and transformers. However, if each port is connected to a voltage source and series resistor  $r_n$ , the claim is that any linear, time-invariant, solvable  $N$ -port is characterized by its  $N \times N$  *scattering matrix*  $S$  [5], [6], [20]. Specialized to the 2-port in Figure 1, the scattering matrix maps

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = S\mathbf{a},$$

where the *incident signal*

$$\mathbf{a} = (R_0^{-1/2}\mathbf{v} + R_0^{1/2}\mathbf{i})/2$$

and the *reflected signal*

$$\mathbf{b} = (R_0^{-1/2}\mathbf{v} - R_0^{1/2}\mathbf{i})/2$$

are computed from the voltage  $\mathbf{v}$  and current  $\mathbf{i}$  via the *normalizing matrix*

$$R_0 = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}.$$

No loss of generality is incurred by taking  $R_0$  as the identity matrix:  $R_0 = I_2$ .

The power  $P$  consumed by the 2-port is provided by Balabanian and Bickart [6, pages 241–242]:

$$P = \Re[\mathbf{v}^H \mathbf{i}] = \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 = \mathbf{a}^H (I_2 - S^H S) \mathbf{a}. \quad (3)$$

If the 2-port consumes no power ( $P = 0$ ) for all its voltage and current pairs, the 2-port is *lossless*. By Equation 3, a 2-port is lossless if and only if its scattering matrix  $S$  is unitary:  $S^H S = I_2$ . How do dilations fit into circuit theory? If Port 2 is terminated in resistor  $r_2$ , then the reflectance  $s_1$  looking into Port 1 is  $s_1 = s_{11}$ . The *circuit synthesis* problem is the converse: given the reflectance  $s_1$ , find all lossless 2-ports

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \text{ with } s_1 = s_{11}.$$

Thus, *circuit synthesis is a problem in dilation theory* [21], [20], [28].

Closely related to the scattering matrix is the *chain scattering matrix*  $\Theta$  [18, page 148]:

$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \Theta \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}.$$

When multiple 2-ports are connected in a chain as in Figure 2, the chain scattering matrix of the chain is product of the individual chain scattering matrices.

When  $S$  is unitary, then  $\Theta$  is a  $J$ -unitary matrix [26], [25], [21]:

$$\Theta^H J \Theta = J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, the lossless 2-ports provide excellent examples of unitary and  $J$ -unitary matrices. The mappings between the scattering matrices and the chain matrices are provided by Hasler and Neiryneck [18]:

$$S \mapsto s_{21}^{-1} \begin{bmatrix} -\det[S] & s_{11} \\ -s_{22} & 1 \end{bmatrix} = \Theta \mapsto \theta_{22}^{-1} \begin{bmatrix} \theta_{12} & \det[\Theta] \\ 1 & -\theta_{21} \end{bmatrix} = S. \quad (4)$$



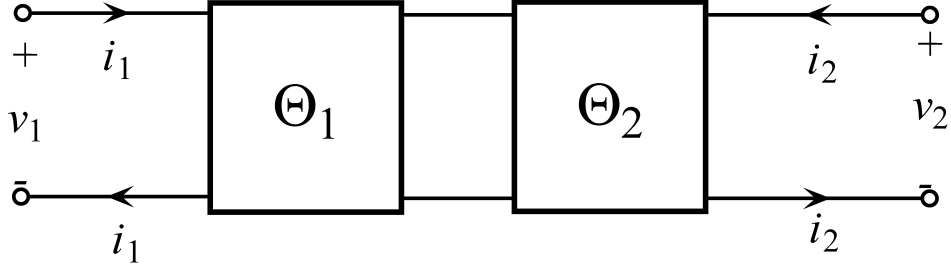


Figure 2: Chain of 2-ports has chain scattering matrix  $\Theta = \Theta_1 \Theta_2$ .

Although the 2-port has a scattering matrix, it admits a chain scattering matrix only if  $s_{21}$  is invertible. These notions generalize to  $N$ -ports. A matrix  $\Theta$  is  $J$ -unitary when

$$\Theta^H J \Theta = J := \begin{bmatrix} I_M & 0 \\ 0 & -I_N \end{bmatrix}.$$

The collection of all such  $J$ -unitary matrices is denoted as  $\mathcal{U}(M, N)$ . Equation 4 generalizes to the map  $\mathcal{S} : \mathcal{U}(M, N) \rightarrow \mathcal{U}(M + N)$  that takes a chain scattering matrix to its corresponding scattering matrix [21], [8]:

$$\mathcal{S}[\Theta] = \begin{bmatrix} \Theta_{12} \Theta_{22}^{-1} & \Theta_{11} - \Theta_{12} \Theta_{22}^{-1} \Theta_{21} \\ \Theta_{22}^{-1} & -\Theta_{22}^{-1} \Theta_{21} \end{bmatrix}. \quad (5)$$

The matrix map  $\mathcal{S}$  is well-defined because

$$J = \begin{bmatrix} I_M & 0 \\ 0 & -I_N \end{bmatrix} = \begin{bmatrix} \Theta_{11}^H \Theta_{11} - \Theta_{21}^H \Theta_{21} & \Theta_{11}^H \Theta_{12} - \Theta_{21}^H \Theta_{22} \\ \Theta_{12}^H \Theta_{11} - \Theta_{22}^H \Theta_{21} & \Theta_{12}^H \Theta_{12} - \Theta_{22}^H \Theta_{22} \end{bmatrix} = \Theta^H J \Theta$$

forces both  $\Theta_{11} \geq I_M$  and  $\Theta_{22} \geq I_N$ . The matrix map  $\mathcal{S}$  turns the CSD and unitary dilations into a hyperbolic CSD and  $J$ -unitary dilations.

## 7 Cosh-Sinh Decomposition

The canonical example of a hyperbolic matrix is given by Mendes and Ruas [26]:

$$H = \begin{bmatrix} \cosh(\psi) & \sinh(\psi) \\ \sinh(\psi) & \cosh(\psi) \end{bmatrix} \in \mathcal{U}(1, 1).$$

This example is a special case of the HCSD.

**Corollary 4 (HCSD)** *Let  $H \in \mathcal{U}(M, N)$  with  $M \leq N$ . Then there are unitary matrices  $U_{11}, V_{11} \in \mathcal{U}(M)$  and  $U_{22}, V_{22} \in \mathcal{U}(N)$  such that*

$$H = \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} \left[ \begin{array}{c|c} \cosh(\Psi) & \sinh(\Psi) & 0 \\ \hline \sinh(\Psi) & \cosh(\Psi) & 0 \\ 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}^H,$$

where  $\Psi = \text{diag}(\psi_1, \dots, \psi_M)$  for  $\psi_m \geq 0$ .

**Proof:** If  $H$  is  $J$ -unitary, Equation 5 makes

$$W = \mathcal{S}[H] = \begin{bmatrix} H_{12}H_{22}^{-1} & H_{11} - H_{12}H_{22}^{-1}H_{21} \\ H_{22}^{-1} & -H_{22}^{-1}H_{21} \end{bmatrix}$$

unitary. To preserve the block structure, apply the CSD as follows:

$$\begin{bmatrix} W_{12} & W_{11} \\ W_{22} & W_{21} \end{bmatrix} = \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} \left[ \begin{array}{c|c} \cos(\Theta) & -\sin(\Theta) & 0 \\ \hline \sin(\Theta) & \cos(\Theta) & 0 \\ 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}^H,$$

where  $\Theta = \text{diag}(\theta_1, \dots, \theta_M)$  for  $0 \leq \theta_m < \pi/2$ . Substitution back into  $H$  gives

$$H = \begin{bmatrix} U_{11} & 0 \\ 0 & -V_{22} \end{bmatrix} \left[ \begin{array}{c|c} \cos(\Theta)^{-1} & \tan(\Theta) & 0 \\ \hline \tan(\Theta) & \cos(\Theta)^{-1} & 0 \\ 0 & 0 & I_{N-M} \end{array} \right] \begin{bmatrix} V_{11} & 0 \\ 0 & -U_{22} \end{bmatrix}^H.$$

Relabeling produces the hyperbolic CS decomposition. ///

Just as the CSD parameterizes the unitary dilations, the HCSD parameterizes the hyperbolic dilations.

## 8 Parameterizing Hyperbolic Dilations

A hyperbolic dilation can be defined in several ways. One approach starts with matrix  $A \in \mathbb{C}^{M \times M}$  such that  $A^H A \geq I_M$ . The problem is to find matrices  $B$ ,  $C$ , and  $D$  such that

$$H_A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is  $J$ -unitary. If  $N \geq M$  and  $A$  has the SVD

$$A = U_{11} \cosh(\Psi) V_{11}^H,$$

then the HCSD and Corollary 1 arguments give that all hyperbolic dilations of  $A$  with  $D \sim N \times N$  have the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} \left[ \begin{array}{c|c} \cosh(\Psi) & \sinh(\Psi) \\ \sinh(\Psi) & \cosh(\Psi) \end{array} \begin{array}{c} 0 \\ 0 \\ I_{N-M} \end{array} \right] \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}^H,$$

where  $U_{22}, V_{22} \in \mathcal{U}(N)$ .

If  $A$  has the SVD

$$A = U_{11} \begin{bmatrix} I_L & 0 \\ 0 & \cosh(\Psi) \end{bmatrix} V_{11}^H,$$

with  $0 < \psi_1 \leq \dots \leq \psi_K$ , then Corollary 2 arguments ensure that the smallest hyperbolic dilations of  $A$  have the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix} \left[ \begin{array}{cc|c} I_L & 0 & 0 \\ 0 & \cosh(\Psi) & \sinh(\Psi) \\ 0 & \sinh(\Psi) & \cosh(\Psi) \end{array} \right] \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}^H,$$

where  $U_{22}, V_{22} \in \mathcal{U}(K)$ .

These hyperbolic dilations are obtained by mapping the hyperbolic matrix back to a unitary matrix, dilating, and mapping back to a hyperbolic dilation. Moreover, just as the CSD generalizes to the hyperbolic CSD, other decompositions of linear algebra admit hyperbolic counterparts. Array processing problems led to a hyperbolic SVD [29], a hyperbolic URV decomposition [35], and a hyperbolic approach to Kalman filtering [16]. The general principle is that the dilations and decompositions obtained for one class of matrices can map into another class of matrices.

Implementing these decompositions in VLSI has been a research topic at the Institute for Network Theory and Circuit Design, Technical University, Munchen, Germany [11]. Diepold and Pauli [9] started with the Schur Decomposition of indefinite matrices. They realized this decomposition was part of the more general problem of embedding a passive matrix in a lossless matrix [10]. They subsequently found a group-theoretic approach that organizes this embedding problem in a signal-processing context that admits hardware solutions [12].

## 9 Antenna-Matching Applications

Figure 3 shows a  $180^\circ$  hybrid chained to a double ferrite antenna and loaded with a lossless 2-port. The hybrid and the antenna are 4-ports. The lossless 2-port is the designable part of this system. The ports are connected by lines that represent the two wires that attach to the two terminals that constitute a port.

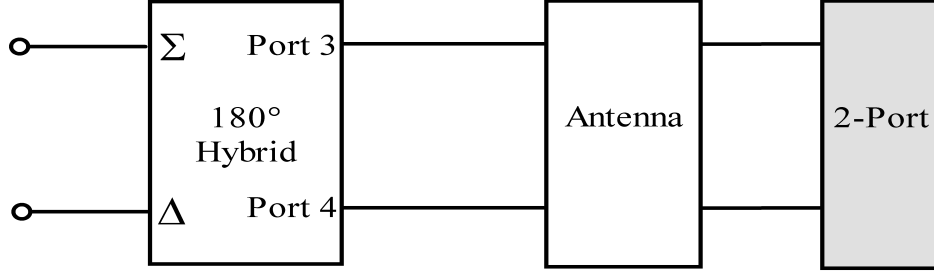


Figure 3: Hybrid-Antenna chain loaded with lossless 2-port (solid).

When the lossless 2-port terminates the antenna, the resulting hybrid-antenna 4-port is converted to a 2-port with scattering matrix

$$S_T = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}.$$

The design goal is find a lossless 2-port that forces  $s_{11}$  and  $s_{12}$  to be small. The parameterizations of the lossless 2-port provide performance bounds. For this particular antenna under consideration, these performance bounds show that this 2-port loading cannot simultaneously force  $s_{11}$  and  $s_{12}$  to be small. Rather than waste time trying to load this antenna, the engineer should look for other antennas that are amenable to 2-port loading.

The hybrid has ideal scattering matrix [31, Equation 7.101]:

$$S_{H,0} = -\frac{j}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

However,  $S_{H,0}$  implies that the ports are indexed as shown in the right side of Figure 4. To chain the hybrid to the antenna, the ports are numbered using the hybrid on the left side of Figure 4.

The renumbered hybrid has scattering matrix [24, Equation 1]:

$$S_H = -\frac{j}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

This scattering matrix  $S_H$  has chain matrix:

$$\Theta_H = -\frac{j}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

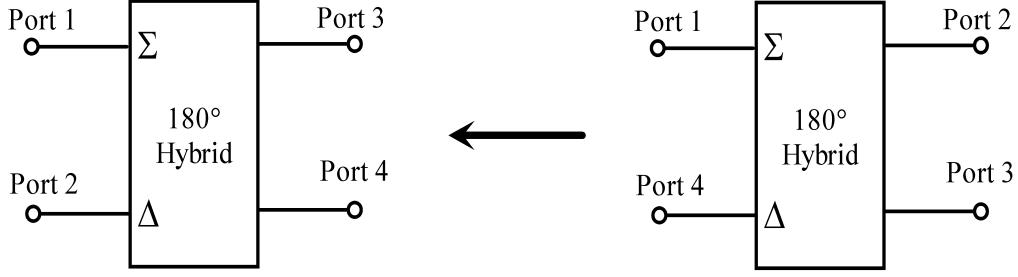


Figure 4: Hybrid port numbering.

For reference, the mapping between the scattering matrix  $S$  and its chain scattering matrix  $\Theta$  is provided by Kimura [25, Equations 4.11, 4.12]:

$$S = \begin{bmatrix} \Theta_{12}\Theta_{22}^{-1} & \Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21} \\ \Theta_{22}^{-1} & -\Theta_{22}^{-1}\Theta_{21} \end{bmatrix}$$

and [25, Equation 4.5]:

$$\Theta = \begin{bmatrix} S_{12} - S_{11}S_{21}^{-1}S_{22} & S_{11}S_{21}^{-1} \\ -S_{21}^{-1}S_{22} & S_{21}^{-1} \end{bmatrix}.$$

Figure 5 plots the scattering matrix  $S_A$  of the Double Ferrite Antenna. Each element of the antenna's scattering matrix

$$S_A = \begin{bmatrix} s_{A,11} & s_{A,21} & s_{A,31} & s_{A,41} \\ s_{A,12} & s_{A,22} & s_{A,32} & s_{A,42} \\ s_{A,13} & s_{A,23} & s_{A,33} & s_{A,43} \\ s_{A,14} & s_{A,24} & s_{A,34} & s_{A,44} \end{bmatrix}$$

is a complex-valued function of frequency  $s_{A,mn}(j2\pi f)$  for  $2 < f < 10$  MHz in the complex unit disk.

Although the scattering matrix  $S_A$  is a  $4 \times 4$  matrix,  $S_A$  is both symmetric,

$$S_A = S_A^T,$$

and centro-symmetric (symmetric across the cross-diagonal),

$$S_A = RS_A^TR; \quad R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

In addition,  $S_A$  is constant along the diagonal and the cross-diagonal. Consequently,  $S_A$  can have no more than four independent functions.

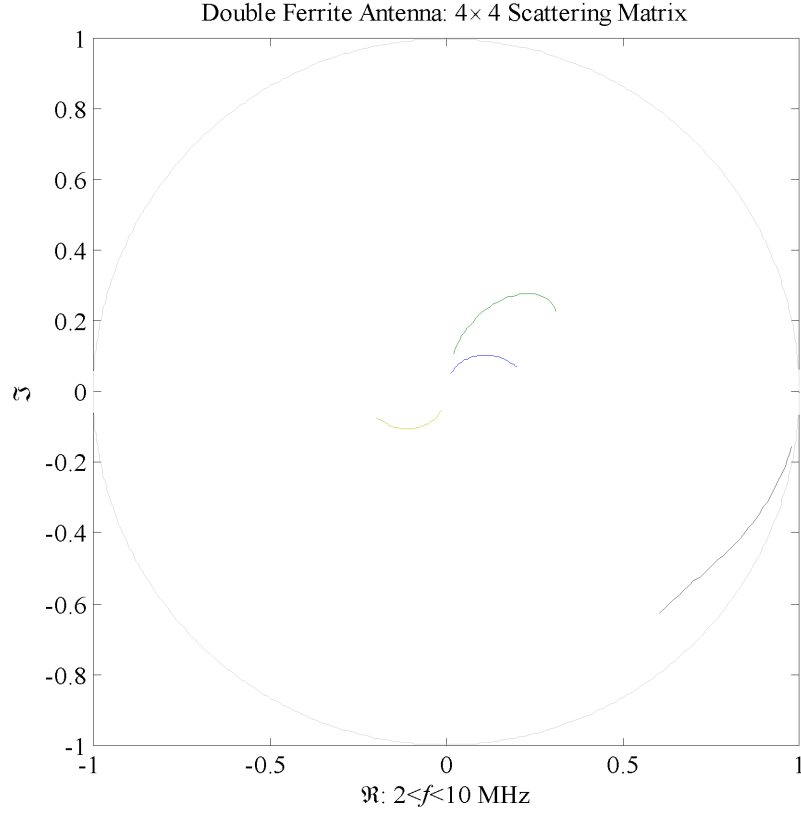


Figure 5: Scattering matrix of Double Ferrite Antenna.

If  $\Theta_A$  denotes the chain scattering matrix of the antenna, the hybrid-antenna chain has scattering matrix computed from the product of the chain matrices:

$$S \Longleftrightarrow \Theta_H \Theta_A.$$

Figure 6 plots the 16 complex-valued functions. The hybrid is destroying some of the antenna's symmetry.

Partition the scattering matrix of the 4-port into  $2 \times 2$  blocks:

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

Let the 4-port be terminated in the lossless 2-port with  $2 \times 2$  scattering matrix  $S_L$ . The resulting 2-port  $S_T$  is obtained by looking into Ports 1 and 2 of the Hybrid while Ports 3 and 4 of the antenna are terminated in the lossless 2-port  $S_L$ :

$$S_T = S_{11} + S_{12}S_L(I - S_{22}S_L)^{-1}S_{21} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}.$$

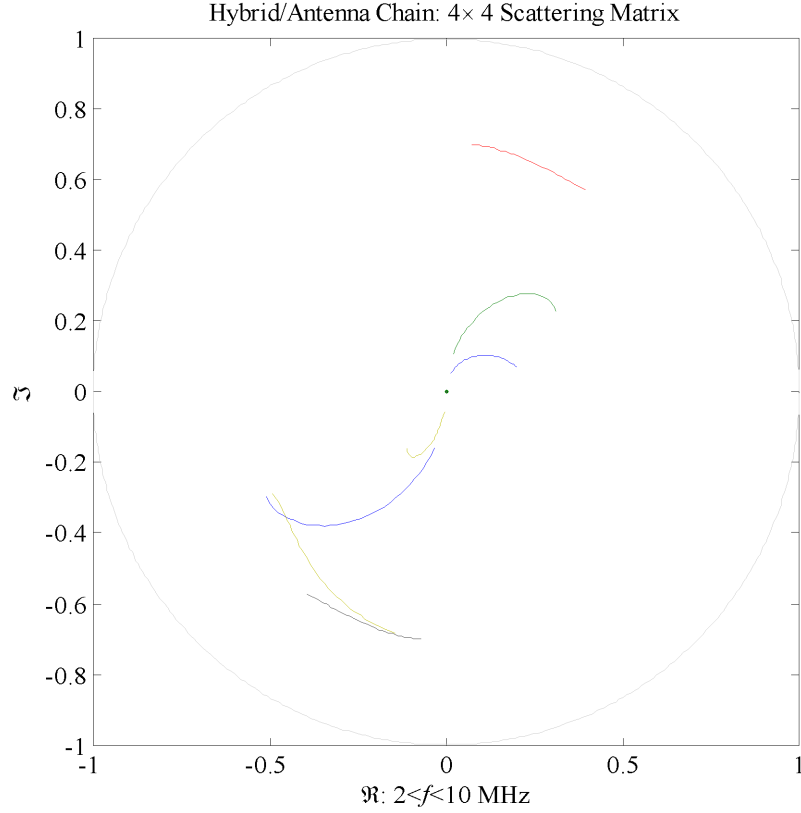


Figure 6: Scattering matrix of Hybrid-Antenna chain.

Observe that  $S$  is really a function of the lossless load and frequency:

$$S(S_L; j\omega) \begin{bmatrix} s_{11}(S_L; j2\pi f) & s_{12}(S_L; j2\pi f) \\ s_{21}(S_L; j2\pi f) & s_{22}(S_L; j2\pi f) \end{bmatrix}.$$

The design goal is to simultaneously minimize  $s_{11}$  and  $s_{21}$ . One multi-objective function is the worst performance at any frequency:

$$\gamma(S_L) = \begin{bmatrix} \|s_{11}(S_L)\|_\infty \\ \|s_{21}(S_L)\|_\infty \end{bmatrix} = \begin{bmatrix} \max\{|s_{11}(S_L; j2\pi f)| : 2 < f < 10\} \\ \max\{|s_{21}(S_L; j2\pi f)| : 2 < f < 10\} \end{bmatrix}.$$

If  $\mathcal{U}$  denotes a class of lossless available to the designer, the multiobjective optimization problem is

$$\min\{\gamma(S_L) : S_L \in \mathcal{U}\}.$$

A lower bound on this performance can be obtained by fixing a frequency:

$$\gamma(S_L; f_0) = \begin{bmatrix} |s_{11}(S_L; j2\pi f_0)| \\ |s_{21}(S_L; j2\pi f_0)| \end{bmatrix} \leq \gamma(S_L).$$

Although  $S_L(j2\pi f_0)$  is a unitary matrix, not all unitary matrices may be parameterized by this  $S_L$ . That is,

$$\{S_L(j2\pi f_0) : S_L \in \mathcal{U}\} \subseteq \mathcal{U}(2).$$

This set inclusion forces the inequality:

$$\min\{\gamma(S, f_0) : S \in \mathcal{U}(2)\} \leq \gamma(S_L; f_0) \leq \gamma(S_L).$$

Consequently, a lower bound on the matching performance may be obtained by fixing the frequency and sweeping over the  $2 \times 2$  unitary matrices of  $\mathcal{U}(2)$ . These unitary matrices can be parameterized by Corollary 1—any  $S \in \mathcal{U}(2)$  has the representation

$$S = \begin{bmatrix} e^{j\phi_{11}} & 0 \\ 0 & e^{j\phi_{22}} \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} e^{j\psi_{11}} & 0 \\ 0 & e^{j\psi_{22}} \end{bmatrix}.$$

Figure 7 plots  $\gamma(S, f_0)$  over a dense sampling of  $S \in \mathcal{U}(2)$ . The plot shows that it is impossible to make  $s_{11}$  and  $s_{21}$  simultaneously small with lossless loading at single frequency. The performance can only be worse over the frequency band. Consequently, Figure 7 tells the antenna engineer not to waste time with any lossless loading design for this antenna. Rather, the antenna must be redesigned for better performance.

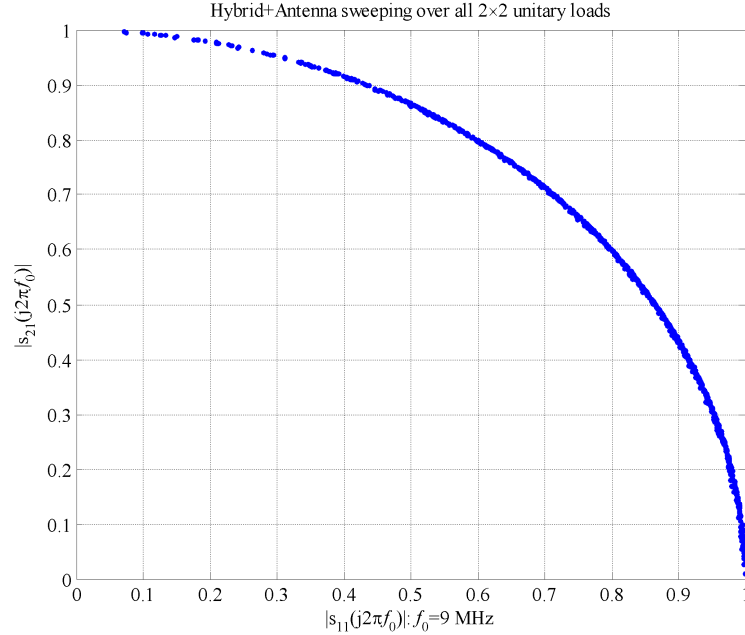


Figure 7: Lower performance bound of lossless loading.



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